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Some comments on turbulence modelling

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ABSTRACT

Various properties of turbulence are discussed with a perspective of their influence on turbulence modelling (TM). The paper considers the relation between spectral properties and TM, the inherent errors associated with TM, the distinction between modelling and numerical errors, local and global processes in turbulence development, the influence of dimensionality on applicability of various models, the potential of LES, and wall proximity corrections.

The emphasis is on understanding when and why TM works, not on demonstrative results.

1. Introduction

Turbulence has been defined often as a 3D unsteady random rotational motion. The turbulent flow field is composed of many structures (or eddies) of various scales. Kolmogorov [1] hypothesized that a small eddy range exists, where dissipation of mechanical energy to thermal energy occurs. It follows (e.g. Tennekes and Lumley [2]) that vortex stretching generates energy transfer from the mean flow to large eddies, and then to smaller and smaller eddies, until the eddies are so small that they dissipate by viscosity. This is often referred to as the "cascade" of energy. Spectral measurements of this flow of energy verified the hypothesis and showed that when the Reynolds number is high the spectral domain can be divided to three sub-ranges:

1. The large eddies in which the flow is non viscous. Convection, diffusion and production of turbulence occur in the large eddies. This range is often referred to as energy carrying range.
2. The small eddies are viscous and tend to isotropy. Viscous dissipation occurs in the small eddies.
3. Between the large energy carrying and the small dissipative eddies lies the inertial sub-range. All the energy supplied to this sub-range from the large eddies is transferred by the cascade mechanism to the small eddies range where it is dissipated to thermal energy.

The consequence of the existence of the inertial zone is that the dissipation is a major characteristic of the turbulence in all three regions. The assumption of isotropy of the turbulence in the viscous region allows analytical approaches to turbulence modelling in this region. Moreover, the existence of the inertial region allows simpler modelling of the turbulence in the energy carrying region. Even if the large eddies are solved using LES, the existence of the inertial region facilitates better modelling of the SGS region in LES (in high Reynolds numbers).

Another useful distinction is between local and global processes. Convection and diffusion of turbulence energy transfer turbulent properties from one location to another in space. However, they do not increase or decrease the amount of turbulences in the entire flow field. Therefore convection and diffusion may be considered "global" processes. On the other hand, production and dissipation produce or destroy turbulence locally, and they may be considered "local" processes. Another important process is the transfer of energy between different components of the Reynolds stress by turbulent pressure–strain interaction. This process is local as well. Global processes are always differential, in the sense that their influence can be predicted only by the solution of differential equations. Local processes are algebraic, or at most, differential of lower order. Their influence is usually calculated by algebraic equations (or simple differential equations). This distinction is often used to explain various turbulent mechanisms and to assist in the modelling process.

The aim of the present paper is to see how these ideas may help the developer and user of turbulence models.

2. The limits of turbulence modelling

Usually theoretical treatment of turbulence depends on the assumption that all turbulent fluctuations are governed by the Navier–Stokes equations. This assumption is justified if the continuum hypothesis and the Stokes stress–strain relation hold even for the smallest turbulent structures. The validity of the continuum

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is discussed in many texts. Here we summarize the treatment of Tennekes and Lumley. Their argument is based on estimates of the various scales of turbulence and is summarized in the following table:

	Mean motion	Large turbulent fluctuations	Small turbulent fluctuations	Molecular motion
Length	L	$L_t = k^{3/2}/\varepsilon$	$L_{Kol} = \nu^{3/4}/\varepsilon^{1/4}$	$L_{mfp} = \nu/a$
Time	L/U	$T_t = k/\varepsilon$	$T_{Kol} = \nu^{1/4}/\varepsilon^{1/2}$	
Velocity	U	$q = \sqrt{k}$	$u_{Kol} = \nu^{1/4}\varepsilon^{1/4}$	
Reynolds number	$RE = \frac{UL}{\nu}$	$R_t = \frac{L\sqrt{k}}{\nu} = \frac{k^2}{\nu\varepsilon}$	$R_{Kol} = 1$	
Mach number	$M = U/a$	$M_t = \sqrt{k}/a$		
Normalizers	L, U	k, ε	ν, ε	

where k is the turbulence energy, ε is the dissipation, ν is the kinematic viscosity and a is the speed of sound.

The ratio of the mean free path to the Kolmogoroff length scale has been represented in the form of a Knudsen number:

$$K_n = \frac{L_{mfp}}{L_k} = \frac{\nu\varepsilon^{1/4}}{a\nu^{3/4}} = \frac{\nu^{1/4}\varepsilon^{1/4}}{a} = M_t\frac{\nu^{1/4}\varepsilon^{1/4}}{k^{2/4}} = \frac{M_t}{Re^{1/4}} = \frac{MT_u}{Re^{1/4}} \tag{1}$$

The Knudsen number grows with the Mach number and the turbulence level (T_u) and it decreases with the Reynolds number. Typical numerical values are given in the following table:

Knudsen number of the dissipative scales						
Tu	0.2					
M	0.1	0.2	0.5	1	2	5
Re						
1.E+03	3.56E−03	7.11E−03	1.78E−02	3.56E−02	7.11E−02	1.78E−01
5.E+03	2.38E−03	4.76E−03	1.19E−02	2.38E−02	4.76E−02	1.19E−01
1.E+04	2.00E−03	4.00E−03	1.00E−02	2.00E−02	4.00E−02	1.00E−01
2.E+04	1.68E−03	3.36E−03	8.41E−03	1.68E−02	3.36E−02	8.41E−02
5.E+04	1.34E−03	2.67E−03	6.69E−03	1.34E−02	2.67E−02	6.69E−02
1.E+05	1.12E−03	2.25E−03	5.62E−03	1.12E−02	2.25E−02	5.62E−02
2.E+05	9.46E−04	1.89E−03	4.73E−03	9.46E−03	1.89E−02	4.73E−02
5.E+05	7.52E−04	1.50E−03	3.76E−03	7.52E−03	1.50E−02	3.76E−02
1.E+06	6.32E−04	1.26E−03	3.16E−03	6.32E−03	1.26E−02	3.16E−02
2.E+06	5.32E−04	1.06E−03	2.66E−03	5.32E−03	1.06E−02	2.66E−02
5.E+06	4.23E−04	8.46E−04	2.11E−03	4.23E−03	8.46E−03	2.11E−02
1.E+07	3.56E−04	7.11E−04	1.78E−03	3.56E−03	7.11E−03	1.78E−02
2.E+07	2.99E−04	5.98E−04	1.50E−03	2.99E−03	5.98E−03	1.50E−02
5.E+07	2.38E−04	4.76E−04	1.19E−03	2.38E−03	4.76E−03	1.19E−02
1.E+08	2.00E−04	4.00E−04	1.00E−03	2.00E−03	4.00E−03	1.00E−02

It is apparent that only when the Mach number becomes very large and the Reynolds number becomes small the Knudsen becomes large enough to make continuum assumption problematic. Therefore turbulence may be regarded as a continuum in most cases. When using the Stokes stress–strain relations is justified, the Navier–Stokes equations represent turbulent flow correctly even in the smallest dissipative scales.

Now we need to consider existence and uniqueness of the solution. There is no general proof of existence of solution to the Navier–Stokes equations. However, if we consider a flow which has been observed experimentally we may assume, on the basis of the previous conclusion that the equations do represent turbulent flows, that a solution exists to the actual problem considered. Uniqueness cannot be proven, and similarity to the experiment would be sufficient for the present argument.

The Navier–Stokes equations constitute a set of partial differential equations of mixed character: parabolic for the velocity components (and the temperature if used) and hyperbolic for the pressure or density. The requirements for boundary and initial conditions are known for such systems, and when all the required conditions are specified the problem is well specified. Under-specification results in an infinite number of solutions and over-specification cannot produce any solution. In other words, if we specify all the required conditions no new information is allowed. This conclusion applies to the fluctuating flow, but it must be valid for the mean equations as well.

Here we face a major problem: usually we are interested in the mean properties of the flow field, not the time-dependent fluctuations. However, the averaging process causes loss of information and addition of yet unknown correlations to the mean equations. Obviously these correlations depend on the lost information which is a part of the existing, although not known, solution of the fluctuating Navier–Stokes equations. In other words, the specification of the mean equations lacks some information, but this information is still a part of the solution of the fluctuating equations, which were well defined. Consequently, the only legitimate way to calculate the information lost by the averaging process is from the fluctuating solution which we do not know.

In other words, the full fluctuating time-dependent equations constitute a well-defined system of equations and as such we are not allowed to add any additional conditions or equations to those required to solve the original set of the Navier–Stokes equations

and their initial and boundary conditions. Turbulence modelling is exactly such an addition. Therefore, we do not solve the equations governing the flow field, but a new system with a different solution. Of course this argument applies only to the modelled equations not to exact equations like the exact, un-modelled equations for the turbulent correlations used. As we do not know the exact solution of the fluctuating Navier–Stokes equations we cannot calculate, or even estimate the difference between the solution of the turbulence model and the solution of the Navier–Stokes equations. Therefore we cannot set bounds on the error cause by the turbulence model, and it is always necessary to validate the model, and the validation is relevant only to validated flows.

In this situation turbulence modelling cannot be considered as tool to explore new problems, but as a smart interpolation technique between measurements.

But are measurements exact? Not necessarily, firstly due to experimental errors. But another very important problem is that often the experimental boundary conditions or even problem parameters cannot be reproduced in the laboratory. Thus both numerical (with turbulence models) and experimental studies are required, and if they are carefully checked and compared to one another they can yield reliable results for engineering (and with caution) to general problems.

In conclusion it is suggested that we should humbly realize that there is no such thing as a "fully universal turbulence model", although turbulence models are an essential tool for the practicing engineer until such days when exact numerical solution of the unsteady three-dimensional Navier–Stokes equations become feasible for any Reynolds number and geometry.

3. On computational errors

In general computational results are prone to some types of errors. All types of errors are usually checked by comparison to experimental results. However, it is necessary to identify the exact cause of the error in order to find the cause and rectify it. The common types of error are:

1. Lack of numerical convergence, by insufficient number of steps or iterations. Careful computation overcomes such problems by running to round-off errors. Sometimes it is necessary to use augmented precision to avoid this problem.
2. Numerical instability, disabling a stable solution (this may affect even time-dependent solutions). Such problems may be difficult to identify, and their solution may require some previous experience.
3. Lack of mathematical convergence, when the exact numerical solutions (of say, the discrete equations) is different from the exact solution of the mathematical equations. This type of error is usually caused by insufficient resolution. Identification of such problems can be easily done using the Richardson extrapolation. However, for large and/or complicated problems one may be limited by the computing resources available.
4. Modelling error, caused by the approximations used during the modelling process. This type of error can be identified only after the first three types (numerical and mathematical) errors have been eliminated. This type of error is different from the previous types of error, firstly because it can be identified only after the other possibilities were eliminated, and secondly because there is no standard technique to identify it apart from comparison with experimental data (or reliable computations, e.g. DNS). The easiest way to verify a model is by comparison with experimental data for simple cases which do not require large computer resources. Typically two-dimensional parabolic flows are often used to do this. However, sometimes even a boundary layer solution may be too demanding. The easiest less demanding problems are those of self-similar flow, where only ODEs are solved, and therefore the need for computer resources is minimized. Free shear layers are a possibility here, because they usually reach self-similarity. However, many modelling problems are caused by the influence of walls, which are absent from free shear layers. Recently Wolfshtein [3] suggested to use solutions of self-similar boundary layers as test problems for turbulence modelling.

Usually self-similar solutions of turbulent boundary layers are impossible due to the different scaling methodology of the inner and outer parts of the boundary layer. However, it was found that when the boundary layer is subject to additional constrains (e.g. injection from the wall) a self-similar solution is possible. Combined experimental/numerical research program of such problems may improve the verification process of turbulence models.

4. Characterization of turbulent flows

In this section we examine the suitability of various classes of turbulence models to various classes of problems.

The first classification is between local (often referred to as algebraic) and global models, as explained below. The first turbulence model published must have been the Prandtl Mixing Length (1925). It is usually referred to as an algebraic model, because no PDE is solved in order to enable the closure of the equations. In due course various algebraic models were developed due to their lower demands on computational resources. The ASM model for the ratios of the Reynolds stress to the turbulence energy probably stretched this class of models more than any, as it still requires PDEs for the turbulent energy and dissipation, but not for the Reynolds stresses themselves. Algebraic models are generated by elimination of the transport processes (convection and diffusion) from the PDE. Thus the equation becomes algebraic rather than differential. Consequently the calculated property depends only on local conditions.

The consequence of this "locality" is that the state of a fluid particle in an arbitrary location within the local region does not depend on the history of the particle along its path before it reached its current location. Local modelling is very useful for verification of models and determination of empirical constants required to complete a model.

Another kind of classification is "dimensionality", which is the number of turbulent stresses relevant in any point in the flow field. This classification is relevant to any eddy viscosity model. The classification is based on the fact that the proportionality coefficient required for a general linear relation between two second order tensor (the stress and the strain) is a fourth order tensor. It is therefore clear that a general scalar eddy viscosity cannot be used in a general case. To illustrate the point we consider a steady 2D incompressible flow field. The momentum equations (neglecting viscous diffusion) are then:

$$\begin{aligned}
&\rho\overline{u_i'u_j'} = -\mu_t\left(\frac{\partial\overline{u_i}}{\partial x_j}+\frac{\partial\overline{u_j}}{\partial x_i}\right)+\frac{2}{3}\rho k\delta_{ij}\\
&\rho\overline{u}\frac{\partial\overline{u}}{\partial x}+\rho\overline{v}\frac{\partial\overline{u}}{\partial y} = -\frac{\partial\overline{p}}{\partial x}+\frac{\partial}{\partial x}\left(2\mu_t\frac{\partial\overline{u}}{\partial x}-\frac{2}{3}\rho k\right)+\frac{\partial}{\partial y}\left[\mu_t\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)\right]\\
&\rho\overline{u}\frac{\partial\overline{v}}{\partial x}+\rho\overline{v}\frac{\partial\overline{v}}{\partial y} = -\frac{\partial\overline{p}}{\partial y}+\frac{\partial}{\partial x}\left[\mu_t\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left(2\mu_t\frac{\partial\overline{v}}{\partial y}-\frac{2}{3}\rho k\right)
\end{aligned} \tag{2}$$

Indeed three different turbulent viscosities are required to obtain a reasonable solution. Yet, if the case considered is such that only the y-derivative is required (as happens in all 2D parabolic flows) the problem disappears and we can obtain a scalar eddy viscosity from

the experimental data, and then it is often possible to devise some theoretical model which agrees with the data.

The above argument is valid even for full 2D flows if the condition of a single flux direction is valid at least locally. We shall define such a condition as "pseudo 1D". For instance, in a back-facing step flow, the condition may be met near the two walls, and also in the mixing layer separating the recirculation bubble from the main flow. The condition is not satisfied in the middle of the bubble itself, but the stresses there are usually small and the associated errors in the bubble do not affect the important features of the solution. Similar arguments apply to full 3D cases as well.

In conclusion, we may define four different environments by considering history effects and locality effects, as follows:

	Local	Global
Pseudo 1D	Algebraic eddy viscosity models	Differential eddy viscosity models
Multi-dimensional	ASM models (but with k and epsilon PDEs)	Reynolds stress models

5. DNS and LES

In view of the inherent difficulties to define accuracy and error bounds of turbulence models theoretically, Direct Numerical Simulation gas been proposed. As the Navier–Stokes equations represent the flow field completely, we may solve the NS equations numerically, and then process and study the time-dependent solution using statistical methods. However, it has been recognized that the required computing resources increase steeply with Reynolds number and therefore DNS can be used only for low Reynolds number cases, until available computer power increases dramatically. The reason is the high spread of size of turbulent structures between the large energy carrying structures and the very fine dissipative structures. Very fine discretization of the flow field is required to overcome this spread, with very high demands on computer storage and computer time.

A cure to this problem was proposed by Large Eddy Simulation. As the dissipative eddies are isotropic, and as the inertial region does not cause important changes in the properties of the turbulence, it appears desirable to separate the regions. The properties of the dissipative range can be calculated using a special turbulence model which can be relatively simple due to the isotropy of the range. Then the expensive discretization may be applied to the large structure range, with large savings in computer resources. Unfortunately, this idea is applicable only to cases in which the inertial range exists. This is not the case for low Reynolds numbers, and in particular to the flow in the immediate vicinity of solid walls. In these regions it is often necessary to refine the discretization and the efficiency improvement of LES over DNS deteriorates. Current efforts are directed towards a combination of conventional turbulence modelling near walls coupled with LES away from walls, taking advantage of the fact that the flow near solid walls is parabolic, and as such it is less demanding computationally. It is the opinion of the author that this approach can reduce the inherent theoretical obstacles of turbulence modelling, but it cannot remove them completely.

6. A few comments on boundary layers and wall functions

The concept of wall functions is based on the idea that the flow near the wall is one-dimensional: all properties change only in the wall-normal direction. It follows that the gradients in the directions parallel to the wall vanish. However this requirement is exactly satisfied only in very few cases like Couette flow or pipe and duct flows. In other cases the requirement is satisfied only on the wall, and as the point in question gets further away from the wall the one-dimensionality deteriorates.

Thus the one-dimensionality assumption may be reasonable only in a narrow layer close to the wall. If this layer is thicker than the viscous sub-layer one may use one-dimensional distribution in this layer to bridge the gap between the boundary conditions specified on the wall and any discrete solution obtained numerically. Whether this distribution is obtained experimentally or analytically is irrelevant.

The most common such distribution is the logarithmic law of the wall and its derivatives. The distribution is obtained from the assumption that the one-dimensional layer protrudes into the turbulent region, beyond the viscous sub-layer. Then it is easy to see that the total shear stress is constant for zero pressure gradient flow or linear for flows with pressure gradient. The logarithmic law is then obtained by exact solution of any turbulence model (indeed this is often the most important test for any new model). Similar techniques can be used to obtain exact solution for more complicated cases like flows with pressure gradient, flows with mass injection from the wall, or compressible flows. In particular a logarithmic distribution can be proven if we assume the a boundary layer can be split into three layers: an inner layer normalized by the friction velocity and the viscosity, an outer layer normalized by the free stream velocity, the boundary layer thickness and the skin friction, and an overlap layer where both sets of normalizing quantities are valid. It is easy to show that if such an overlap layer exists, the velocity profile there must be logarithmic.

However, the assumption of uniform or linear shear is even more problematic then that of one-dimensionality near the wall. Thus all such solutions are only approximations. Therefore one cannot disqualify other assumptions. In recent years various authors suggested to use power law velocity profiles. There is no theoretical justification to such profiles, but in view of the many approximations used to obtain the logarithmic profiles (even the boundary layer approximation is exact only at infinite Reynolds number) the power law profile cannot be discarded. Indeed it is not difficult to replace one by the other by fitting the velocity and velocity gradient at any selected point. Both profiles are mathematically poor in the sense that the logarithmic profile is singular at the wall in both the velocity and the velocity gradient, while the power law profile is singular only in the velocity gradient at the wall. Therefore the power law profile may be more convenient for certain applications.

The actual derivation of the wall functions follows from the choice of the velocity profile and is not discussed here. However, one should bear in mind that the wall functions approach and its origin, the assumption of one-dimensionality, are exact only in infinite Reynolds numbers. For low Reynolds numbers various empirical corrections are applied near the wall. This is true also for the "down to the wall" solution where wall functions are not used. Instead the model equations are modified near the walk so that their solution agrees with the one-dimensional solution.

7. Conclusions

The major worry about turbulence modelling is that one cannot estimate the errors of the computer results. Indeed, it is impossible even to define error bounds. However, there is no better way to calculate turbulent flows unless we deal with very low Reynolds numbers and simple geometries. This may change in the future with improvements in computer technology, but any predictions here may be very risky. It is true that the "Moore law" for increase

in computing power was successful for many years, but today it is often claimed that future advancement in computer technology will depend on the development of entirely new technologies, which are difficult to predict.

One could use experimental data instead, but this is an expensive and slow process, and it has its own problems, like imitation of far field boundary conditions and agreement with all relevant parameters (e.g. it is almost impossible to run a wind tunnel test with the actual Reynolds and Mach numbers).

Another important issue is the identification of numerical versus modelling errors. This is particularly relevant for large problems where mesh refinement may be prohibitively expensive.

What is the answer then? Unfortunately it is still the old traditional answer: one should exercise caution and compare information from various sources: experimental, numerical, theoretical and industrial. In this sense prediction of turbulent flow is still an art. Although it is relatively easy to get a solution to most problems, it is very difficult to assign a reliability index to such a solution. There is still no substitute to thorough professional understanding and experience. However, when these qualities are properly applied the results may be rewarding and useful, to the extent that many, if not most designers would wish to have a numerical solution as a part of their project development.

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